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ON THE PRODUCT OF TWO COMMUTATIVE OPERATORS.

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The aim of this note is to give a very elementary explanation of the order of the product of two commutative operators. Such an explanation seems the more desirable on account of the fact that an error in regard to this matter occurs in one of the best known works on the theory of groups.* We shall first consider the cases when the orders of the two operators (s_1, s_2) are powers of the same prime number (p), the order of s_1 being p^α and that of s_2 being p^β ($\alpha \overline{\gtrless} \beta$).

From the equation $(s_1s_2)^n=s_1^ns_2^n$ and the fact that $s_1^ns_2^n=1$ only when $s_1^n=s_2^{-n}$, it follows that s_1s_2 is of the same order as s_1 whenever $\alpha>\beta$. When $\alpha=\beta$ several cases present themselves: (1) The groups generated by s_1 and s_2 have only identity in common. In this case it follows from the given equations that s_1s_2 is of the same order as s_1, viz., of order p^α. (2) s_1 is a power of s_2; *e. g.* $s_1=s_2^\gamma$. In this case the order of s_1s_2 may be any power of p from p^0 to p^α when $p>2$, and p^0 to $p^{\alpha-1}$ when $p=2$. These orders may be obtained by assigning the following values to γ:

$$p^\alpha-1,\ p^{\alpha-1}-1,\ p^{\alpha-2}-1,\ \ldots .p-1,\ 1.$$

In general, let $s_1p^{\alpha_1}$ be the first power of s_1 which is also a power of s_2, so that $s_1p^{\alpha_1}=s_2kp^{\alpha_1}$ (α_1 does not equal α, and k being prime to p and less than $p^{\alpha-\alpha_1}$). It follows from the first sentence of the preceding paragraph that the

*Burnside, *Theory of Groups of a Finite Order*, 1897, page 16.

order of $s_1 s_2$ cannot be less than p^{α_1} nor greater than p^{α}. If we assign to k the following values:

$$p^{\alpha-\alpha_1}-1,\ p^{\alpha-\alpha_1-1}-1,\ p^{\alpha-\alpha_1-2}-1,\ \ldots . p-1,\ 1\ (\alpha_1 \text{ does not equal } \alpha),$$

we observe that $s_1 s_2$ may have for its order any power of p from p^{α_1} to p^{α} when p is odd (or when $p=2$ and $\alpha=\alpha_1$), and from p^{α_1} to $p^{\alpha-1}$ when $p=2$ and $\alpha>\alpha_1$.

These results are expressed by the following

THEOREM. *It is possible to find two commutative operators* (s_1, s_2) *of the same order* (p^{α}) *such that* $s_1 p^{\alpha_1}=s_2 kp^{\alpha_1}$ $(\alpha \overline{\leq} \alpha)$ *and* $s_1 s_2$ *is of order* p^{δ}, *where* δ *can have any value from* α_1 *to* α *when* $p>2$ (*or when* $p=2$ *and* $\alpha=\alpha_1$), *and from* α_1 *to* $\alpha-1$ *when* $p=2$ and $\alpha>\alpha_1$.

When the orders of s_1 and s_2 are not powers of the same prime they may be represented by $2^{\alpha_0} p_1{}^{\alpha_1} p_2{}^{\alpha_2} \ldots .$ and $2^{\beta_0} p_1{}^{\beta_1} p_2{}^{\beta_2} \ldots .$ respectively; $p_1, p_2, \ldots .$ being odd prime numbers, and the exponents being positive integers including 0. We may think of s_1 and s_2 as the products of operators of orders 2^{α_0}, $p_1{}^{\alpha_1}$, $p_2{}^{\alpha_2}$, ... and of orders 2^{β_0}, $p_1{}^{\beta_1}$, $p_2{}^{\beta_2}$, ... respectively. $s_1 s_2$ is then the product of all of these operators. Combining the pairs which are powers of the same prime, $s_1 s_2$ may be represented as the product of operators of orders 2^{γ_0}, $p_1{}^{\gamma_1}$, $p_2{}^{\gamma_2}$, ... ($p_x{}^{\gamma_x}$ being the product of the given operators of orders $p_x{}^{\alpha_x}$ and $p_x{}^{\beta_x}$). Since the groups generated by these operators have only identity in common $s_1 s_2$ is of order $2^{\gamma_0} p_1{}^{\gamma_1} p_2{}^{\gamma_2} \ldots .$ We have now reduced this case to one of the earlier ones.

It was observed in the second paragraph that γ_x is equal to the larger of the two numbers α_x and β_x whenever these are different, and that γ_x may be 0 whenever $\alpha_x=\beta_x$. *Hence the minimum order* (m) *of* $s_1 s_2$ *is the product of the highest of the primes which divide one and only one of their orders.** The maximum order (M) of $s_1 s_2$ is clearly the lowest common multiple of their orders. From the given theorem it follows that s_1 and s_2 can be so selected that the order of $s_1 s_2$ is any factor of M which is not less than m.

HYDRAULIC SOLUTION OF AN ALGEBRAIC EQUATION OF THE nth DEGREE.

By DR. ARNOLD EMCH, University of Colorado.

In the January (1901) number of this MONTHLY I have established two methods of extracting the nth root of any positive real number. In conclusion, I proposed to apply the first (hydrostatic) method to the solution of an equation of the form

$$a_0+a_1x+a_2x^2+\ldots .+a_nx^n=0\ldots .(1).$$

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*Mr. Fite first called my attention to this minimum value.